MATHEMATICS
P.G. Sem II (18-20)
Topic – Complex variable (Power series)
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A Series of the form

$$\sum_{n=0}^{\infty} a_n z^n \quad \text{or} \quad \sum_{n=0}^{\infty} a_n (z-a)^n$$

is called a power series where a_n and a are complex constants and z is a complex variable

If $z-a = \xi$ (complex variable)

Then $\sum_{n=0}^{\infty} a_n (z-a)^n = \sum_{n=0}^{\infty} a_n \xi^n$ which is in the first form. So first and second both are in the same form.

Absolute convergence of $\sum a_n z^n$:

The power series $\sum a_n z^n$ is said to be absolutely convergent if the series $\sum |a_n| |z|^n$ is convergent.

The power series $\sum a_n z^n$ is said to be conditionally convergent if $\sum a_n z^n$ is convergent, but $\sum |a_n| |z|^n$ is not convergent

Radius of convergence of power series

- Now we have $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} |z| < 1$

Then taking $\lim_{n\to\infty} |a_n|^{1/n} = \frac{1}{R}$

Then we get $\frac{|z|}{R} < 1$ i.e. $|z| < R$

Thus $\sum a_n z^n$ is ~~absolut~~ convergent when $|z| < R$ and divergent when $|z| > R$

So corresponding to every power series there exists a positive (non-negative) real number R such that if the series is convergent then $|z| < R$ and if the series is divergent then $|z| > R$.

Now we consider a circle with centre at the origin and of radius R then :

I. The power series is convergent for every z within the circle and

II. The power series is divergent for every z outside the circle.

Such a circle ~~Z~~ is called circle of convergence and its radius is called radius of convergence of the power series $\sum a_n z^n$.

Now three cases will arise :

(I) When $R = 0$.

In this case the series is convergent only when $Z = 0$

(II) When R is finite.

In this case the series is convergent at every point within the circle and divergent at every point outside the circle

(III) When R is infinite

In this case the series is convergent for every point z.

Cauchy Hadamard Theorem

For every power series $\sum_{n=0}^{\infty} a_n z^n$, there exists a number R such that $0 \leq R \leq \infty$ with the following properties

(I) The series converges absolutely for every Z such that $|z| < R$

(II) The series diverges if $|z| > R$

Proof: we write the previous topic i.e. radius of convergence of power series.

Abel's Theorem

If the power series $\sum a_n z^n$ converges for a particular value Z_0 of Z, then it converges absolutely for all values of Z for which $|Z| < |Z_0|$

Proof

Let the series $\sum a_n z_0^n$ converges then it's n^{th} term $a_n z_0^n$ must tend to 0 when $n \to \infty$

i.e. $\lim_{n \to \infty} a_n z_0^n = 0$.

So there exists a positive number M such that

$|a_n z_0^n| \leq M$ for all n.

i.e. $|a_n z^n| \leq M \left|\frac{z}{z_0}\right|^n$

It is given that $|z| < |z_0|$ so $\left|\frac{z}{z_0}\right| < 1$ and therefore the geometric series $\sum_{n=0}^{\infty} \left|\frac{z}{z_0}\right|^n$ is convergent.

Hence by comparision test the series $\sum |a_n z^n|$ is convergent for all z for which $|z| < |z_0|$

Therefore the power series $\sum a_n z^n$ is absolutely convergent for all z satisfying

$|z| < |z_0|$.